Effective theory of NN interactions in a separable representation

B. Krippa and B.L. G. Bakker

Department of Physics and Astronomy, Free University, Amsterdam,
De Boelelaan 1081, 1081 HV Amsterdam.

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Abstract

We consider the effective field theory of the NN system in a separable representation. The pionic part of the effective potential is included nonperturbatively and approximated by a separable potential. The use of a separable representation allows for the explicit solution of the Lippmann-Schwinger equation and a consistent renormalization procedure. The phase shifts in the $^1S_0$ channel are calculated to subleading order.

1 Introduction

During the last few years the effective field theory (EFT) has extensively been used for the study of the NN interactions. The activity in this field was inspired by the Weinberg proposal [1] that the EFT approach could be useful in low energy nuclear physics. Since Weinberg's original paper many aspects of this problem have been discussed [2].

Contrary to more phenomenological models of hadron interactions EFT allows for systematic expansion of the scattering amplitude order by order and the possibility to estimate a priory the anticipated errors at each order of the expansion using power counting rules. However, while applied to the two-nucleon systems, EFT encounters a serious difficulty which is due to the existence of the extremely large the S-wave scattering length (compared to the pion Compton wavelength). Thus, it turns out that the EFT description of the NN forces must be nonperturbative to incorporate this large scale. In the original work [1] Weinberg proposed to

apply counting rules to the irreducible diagrams in order to construct the effective potential which is to be iterated in the Lippmann-Schwinger (LS) equation. Immediately one can see a complication . The corresponding effective potential is highly singular. The origin of this singularity is the local nature of the nucleon-nucleon coupling. In order to obtain finite physical observables one needs to carry out the procedure of regularization and renormalization. The issue of renormalization is much more involved in the case of the nucleon-nucleon interaction as compared to the standard perturbative situation where the renormalization can be carried out for the set of individual Feynman diagrams, using the standard textbook methods. For the problem in hand, nonperturbative renormalization is required so that at every order the divergences of the whole nonperturbative amplitude must be subtracted.

A somewhat different way to construct the EFT of the NN-forces has been proposed some time ago by Kaplan et al. [3]. The idea was to sum up a certain subclass of the leading order diagrams, given by the lowest order contact interactions. The rest, including the higher order contact interactions and graphs with pions, can then be treated perturbatively. This approach is systematic, chirally symmetric, and is formulated in such a way that chiral counting rules can be applied directly to the nucleon-nucleon scattering amplitude. The leading non-perturbative amplitude can be calculated in an analytic form, allowing for the renormalization to be carried out in explicit and transparent way. The renormalization of the perturbative corrections can be performed using the standard methods of dealing with divergencies of Feynman diagrams. However, the perturbative "pionic part" of this approach seems to show rather slow convergence in some particular channels [4], making the practical use of this approach somewhat problematic.

In the Weinberg approach pion effects are treated to all order. At very low energies, when pion degrees of freedom can safely be integrated out, the scattering amplitude can be derived analytically and so no problem with the renormalization arises. In the more general case of a potential consisting of the contact terms and a long-range one pion exchange (OPEP) contribution, the analytic solution of the three-dimensional LS equation is no longer possible and the problem must be treated numerically. However, it is not at all clear how to carry out the renormalization in such a situation. One notes that it is not enough to regularize the integral part of the LS equation by imposing a simple cut-off or using form-factors. In this case one is still left with the bare couplings and the physical amplitude may strongly depend on the value of the cut-off parameter. It contradicts the renormalization group requirements, according to which [5] the physical NN-amplitude must be cut-off independent (at least up to the order one is dealing with). To remove the unwanted cut-off dependence one needs to switch to renormalized effective couplings. However, this is difficult to implement in the situation where the analytical solution is not known.

In this paper we propose an approximate method of how to carry out the renormalization if an exact solution of the LS equation is not possible. Namely, we

propose to use the approximate analytical solution, which can be obtained if we represent the pionic part of the effective Lagrangian by a sum of separable terms. In this case the integral equation can be transformed into a matrix equation and an analytical solution becomes possible. Then the renormalization can be carried out by subtracting the loop integrals at some fixed kinematical point $p^2 = -\mu^2$ and by replacing the bare constants with the running ones, depending on the point of subtraction.

2   Model

We start from the standard nonrelativistic effective Lagrangian

$$\mathcal{L} = N^\dagger i \partial_t N - N^\dagger \frac{\nabla^2}{2M} N - \frac{1}{2}C(N^\dagger N)^2 - \frac{1}{2}C_2(N^\dagger \nabla^2 N)(N^\dagger N) + \mathcal{L}_\pi + h.c. + \dots \quad (1)$$

Here $\mathcal{L}_\pi$ is the "pionic" part of the effective chiral Lagrangian. This Lagrangian leads to the following effective potential for the $^1S_0$ NN scattering [1]

$$V(\vec{p}, \vec{p}') = C' + C_2(\vec{p}^2 + \vec{p}'^2) + V_\pi(\vec{p}, \vec{p}'), \quad (2)$$

where

$$C' = C + \frac{g_A^2}{2f_\pi^2}; \qquad V_\pi(\vec{p}, \vec{p}') = -\frac{\alpha_\pi}{\vec{q}^2 + m_\pi^2}; \qquad \alpha_\pi = \frac{g_A^2 m_\pi^2}{2f_\pi^2}, \quad (3)$$

$\vec{q} = \vec{p} - \vec{p}'$, $g_A = 1.25$ and $f_\pi = 132$ MeV are the axial and pion decay constant respectively. As mentioned above the consistent numerical realization of the renormalization program in the nonperturbative situation is a very difficult task (see for example Ref.[6]) therefore we adopt the strategy of an approximate analytic solution of the LS equation allowing for the explicit realization of the renormalization procedure. To achieve this goal we represent the OPEP contribution by a sum of the separable terms. As we shall henceforth limit our discussion to S-waves only, the matrix elements are functions of the magnitudes of the momenta only. We write

$$V_\pi(p, p') = \sum_{j=1}^n \alpha_j \eta_j(p) \eta_j(p'). \quad (4)$$

One notes that, in principle, $V_\pi(p, p')$ can be parametrized with arbitrary accuracy but in this short letter we rather would like to emphasize the issues related to renormalization in the effective description of the NN interaction. So in practice we retain only one term in a separable expansion. It turned out to be enough to illustrate the main features of our approach. Of course, this is a quite crude description of the OPEP, which approximate the exact pionic part of the effective Lagrangian with an average error about 10-12% in the momentum region 0.4

$\mathrm{fm}^{-1} < p < 1.4 \ \mathrm{fm}^{-1}$. We postpone the detailed analysis of the NN observables in the different partial waves and spin-isospin channels until future publication.

After the separable approximation is substituted, the effective potential can be represented in the following matrix form

$$V^{\mathrm{eff}}(p, p') = \sum_{ij} g_i(p) M_{ij}(p) g_j(p'), \tag{5}$$

where

$$g_i(p) = \begin{pmatrix} 1 \\ p^2 \\ \eta_1(p) \end{pmatrix} \quad \text{and} \quad M_{ij} = \begin{pmatrix} C' & C_2 & 0 \\ C_2 & 0 & 0 \\ 0 & 0 & \alpha_1 \end{pmatrix}. \tag{6}$$

The solution of the LS equation can be represented as

$$T(p, p'; E) = g_i(p) \tau_{ij}(E) g_j(p'). \tag{7}$$

We denote τ the 3×3 matrix containing the loop integrals $I_{ij}(E)$, given by

$$\tau(E) = [1 - M \, I(E)]^{-1} M, \tag{8}$$

where

$$I_{ij}(E) = \int_0^{\infty} \frac{dq q^2}{2\pi^2} \frac{g_i(q) g_j(q)}{E + i\epsilon - E(q)}. \tag{9}$$

The matrix $\tau(E)$ contains convergent and divergent integrals so regularization and renormalization must be carried out. We use the subtraction scheme smilar tho the one suggested suggested in [7]. Namely, all loop integrals are subtracted at some kinematical point $p^2 = -\mu^2$. The renormalized T-matrix is

$$T^{\mathrm{Reg}}(p, p'; E) = g_i(p) \, \tau_{ij}^{\mathrm{Reg}}(E) \, g_j(p'). \tag{10}$$

In the following we will omit the superscript "Reg" implying that we always work with the renormalized amplitude. After renormalization the Low-energy effective constants become dependent on the renormalization point μ. The "μ independence" of the scattering amplitude is provided by the renormalization group (RG) equations. Requiring that $dT/d\mu = 0$ and using the analytic expression for the T-matrix one obtains the following set of RG equations for the leading order coefficient C.

$$\frac{\partial C(\mu)}{\partial \mu} = \frac{C'^2 M}{4\pi} - 2\alpha \eta_1^2(p) C' \frac{M}{4\pi}. \tag{11}$$

Neglecting the term with the form-factors $g_i(p)$ we arrive at the variant of the RG equations first derived by Kaplan et al. [3] where pions were included perturbatively. In the region where the second term becomes nonnegligible, the pionic effect must be treated in a nonperturbative manner.

3   Numerical Results

We used the exponential form of the separable form-factors to parametrize the one-pion exchange potential

$$\eta_1(p) = \exp(-\beta p). \tag{12}$$

The cut-off parameter β and strengh parameter α are taken to be 0.78 fm and 1.73 fm^2 respectively. The values of the effective constants used to calculate the phase shifts are $C(m_\pi) = -3.2$ fm^2 and $C_2(m_\pi) = 2.5$ fm^4. These values are to be compared with the chiral counting rules, according to which $C_{2n}(\mu) \sim 4\pi/(M\Lambda^n\mu^{n+1})$, where Λ is the scale where chiral perturbation theory breaks down. Assuming $\Lambda \sim 300 - 400$ MeV, one finds that the values of the effective constants are indeed consistent with the counting rules, although somewhat lower than thode obtained in Ref. [3]. One notes that it is hard to compare the effective constants obtained in different regularization schemes, since the coupling is known to be a scheme dependent quantity.

The nonperturbative corrections due to the separable potential with the form-factor $\eta_1(p)$ become noticeable at $p \sim 100$ MeV/c. It agrees with the estimates obtained in [9]. Of course, the precise region where pions become nonperturbative may somehow depend on the concrete form-factors used, but the general tendency of the pion effects to become too strong to be treated perturbatively at $p > 0.5$ fm^{-1} seems to be quite robust, making the whole problem much more complicated.

As already mentioned in this paper we focus on the $^1S_0$ channel and calculate observables up to next-to-leading order. The main goal was to develop a reasonable calculational scheme with consistent renormalization procedure so we retained only one term in the separable expansion of OPEP contribution. Of course, it gives only a crude parametrization of the long-range part of the effective Lagrangian so that our comparison of the theoretical results with the experimental phase shifts has somewhat illustrative character to demonstrate the feasibility of the method proposed. The results obtained are shown in Table 1.

The deviation from the Nijmegen phase shifts [8] is about 12-15% on average in the kinematical region 0.4 fm^{-1} < p < 1.35 fm^{-1}. At lower momenta the pionic effects can either be integrated out or safely treated perturbatively. At larger momenta the next-next-to-leading order corrections like two-pion-exchange or $O(p^4)$ contact terms become more and more important and must be taken into account. The errors of the theoretical analysis are comparable with those introduced by the separable representation of the effective potential, so no significant additional uncertainties are introduced by the loop integration. Therefore one could hope that taking into account a few more terms in the separable expansion of the effective potential will bring the theoretical results into better agreement with the experimental phase shifts. Work in this direction is in progress.

In summary, we analyzed the problem of renormalization in the effective theory of the NN interaction when the perturbative chiral expansion is not valid.

Table 1: Numerical results for the phase shifts $\delta(^1S_0)$ in degrees obtained within the separable approximation to the OPEP compared to the results from the Nijmegen phase-shift analysis. T_{Lab} is given in MeV and p in fm^{-1}.

T_{Lab}	p	$\delta_{\text{sep pot}}$	δ_{Nijm}
20	0.48	46.9	53.6
50	0.77	43.6	40.1
70	0.91	40.1	34.3
90	1.02	34.7	29.1
110	1.13	26,8	24.6
130	1.24	17.6	20.6
150	1.32	10.0	16.9
170	1.41	2.6	13.6

In Weinberg's approach, where pions are treated nonperturbatively, the scattering amplitude can be found only numerically, making the procedure of consistent renormalization difficult to implement. On the other hand, in the approach proposed by Kaplan et al., pions are treated perturbatively, so that renormalization can be carried out in the standard way. The latter approach however, shows rather slow convergence in some channels. The procedure we propose is based on the approximate but nonperturbative treatment of pionic effects based on a separable expansion of the long-range part of the effective potential ans allowing for the renormalization to be carried out in an analytic form. Our method gives a reasonable description of the $^1S_0$ NN phase shifts in the laboratory-energy region up to $T_{\text{Lab}} \sim 140$ MeV.

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